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STABILITY OF THE EQUILIBRIUM OF AN ELASTIC RING IN THE PRESENCE OF RADIAL SHEAR*

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An exact formulation of the non-linear theory of elasticity is used to study the problem of bifurcation of equilibrium in a circular annulus. It is assumed that the inner boundray is rigidly clamped and tangential stresses act on the outer boundary. The material considered is isotropic and incompressible, with elastic Mooney potential. The subcritical state is determined from the exact solution of the problem of shear under finite deformations. After separating the variables in the equations of neutral equilibrium, the problem is reduced to a system of ordinary differential equations. The critical value of the tangential load intensity is determined by numerical methods.

1. Subcritical state. Let us consider an elastic ring $1 \leqslant r \leqslant 1+h$ rigidly clamped along the inner contour and acted upon along the outer contour by a tangential follower load whose intensity vector per unit arc length of the deformed contour has the form

$$\mathbf{P} = \tau\mathbf{k} \times \mathbf{N} \tag{1.1}$$

Here $\mathbf{k}$ is the unit vector orthogonal to the ring plane, $\mathbf{N}$ is the material normal to the deformed contour, and the quantity τ does not vary during the deformation process. Let r, φ $(0 \leqslant \varphi \leqslant 2\pi)$ be the polar coordinates in the undeformed state of the body and R, Φ the polar coordinates of the points of the body after the deformation. We denote the orthonormal vector bases connected with these coordinates by $\mathbf{e}_r, \mathbf{e}_\varphi$ and $\mathbf{e}_R, \mathbf{e}_\Phi$, respectively. We define the shear deformation by means of the following relations:

$$R = R(r); \quad \Phi = f(r) + \varphi \tag{1.2}$$

From (1.2) we obtain

$$\mathbf{C} = \mathbf{e}_r(R'\mathbf{e}_R + Rf'\mathbf{e}_\Phi) + Rr^{-1}\mathbf{e}_\varphi\mathbf{e}_\Phi \tag{1.3}$$

where $\mathbf{C}$ is the deformation gradient and the prime denotes differentiation with respect to r. The condition of incompressibility $\det \mathbf{C} = 1$ yields $R'Rr^{-1} = 1$. Integrating the last relation we obtain $R = r$. Then using (1.3) we can write the following expressions for the Finger and Almansi deformation measures $\mathbf{F}$ and $\mathbf{g} = \mathbf{F}^{-1}$:

$$\mathbf{F} = \mathbf{e}_R\mathbf{e}_R + \psi(\mathbf{e}_R\mathbf{e}_\Phi + \mathbf{e}_\Phi\mathbf{e}_R) + (1+\psi^2)\mathbf{e}_\Phi\mathbf{e}_\Phi \tag{1.4}$$

$$\mathbf{g} = \mathbf{e}_R\mathbf{e}_R(1+\psi^2) + \mathbf{e}_\Phi\mathbf{e}_\Phi - \psi(\mathbf{e}_R\mathbf{e}_\Phi + \mathbf{e}_\Phi\mathbf{e}_R); \quad \psi = Rf'$$

We shall consider a homogeneous ring made of Mooney material. We will write the law of conservation of an isotropic elastic incompressible material in the Finger form /1/

$$\mathbf{T} = \varkappa_1\mathbf{F} - \varkappa_2\mathbf{g} - p\mathbf{E} \tag{1.5}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{E}$ is the unit tensor, p is the pressure in the incompressible material not determinable by the deformation, and $\varkappa_1, \varkappa_2$ are the elastic material constants. Taking (1.3), (1.4) into account we reduce the equations of equilibrium to the form

$$\frac{\partial p}{\partial R} = -2\psi\psi'\varkappa_2 - \frac{\psi^2}{R}(\varkappa_1 + \varkappa_2), \quad \frac{\partial p}{\partial \Phi} = (2\psi + \psi'R)(\varkappa_1 + \varkappa_2) \tag{1.6}$$

and using the condition $\partial^2 p/\partial R\,\partial\Phi = \partial^2 p/\partial\Phi\,\partial R$ we obtain from (1.6)

$$\psi = -2C_1R^{-2} + C_2; \quad C_1, C_2 = \text{const}$$

Let us denote the Cauchy stress tensor components by $\sigma_{RR}, \sigma_{R\Phi}, \sigma_{\Phi\Phi}$. Then the boundary conditions at the ring surface will be

$$\sigma_{RR} = 0, \quad \sigma_{R\Phi} = \tau, \quad R = 1+h; \quad \Phi = \varphi, \; R = 1 \tag{1.7}$$

Since we use here the dimensionless values of the radial variable, the quantity h represents the ratio of the ring thickness to its inner radius. Satisfying the boundary conditions (1.7) and using the formula (1.4), (1.5), we obtain the following expressions for the functions f and p and the components of the tensor $\mathbf{T}$:

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$$f = C_1 (R^{-2} - 1); \quad C_1 = -\frac{\tau (1+h)^2}{2(\varkappa_1 + \varkappa_2)}, \quad C_2 = 0 \tag{1.8}$$
$$p = \varkappa_1 + \varkappa_1 C_1^2 [R^{-4} - (1+h)^{-4}] - \varkappa_2 - \varkappa_2 C_1^2 [3R^{-4} + (1+h)^{-4}]$$
$$\sigma_{RR} = (\varkappa_1 + \varkappa_2)[(1+h)^{-4} - R^{-4}] C_1^2$$
$$\sigma_{R\Phi} = -2R^{-2} (\varkappa_1 + \varkappa_2) C_1; \quad \sigma_{\Phi\Phi} = (1 + 4|C_1^2 R^{-4}) \varkappa_1 - \varkappa_2 - p$$

2. Neutral equilibrium equations. To reduce the bulk of the formulas we shall consider a particular case of the Mooney material, namely a neo-Hookean material. The formulas of Sect.1 are easily transformed to this case by putting $\varkappa_2 = 0$. Let us write the equations of equilibrium linearized to the neighbourhood of the given stress state. We will write them for the incompressible body in the metric of subcritical state as follows /1, 2/:

$$\nabla \cdot \boldsymbol{\Theta} = 0; \quad \boldsymbol{\Theta} = \varkappa_1 \mathbf{F} \cdot \nabla \mathbf{w} + p (\nabla \mathbf{w})^T + q\mathbf{E} \tag{2.1}$$
$$\nabla \cdot \mathbf{w} = 0; \quad \nabla = \mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\Phi \frac{\partial}{R \partial \Phi}$$

where $\mathbf{w}$ is the additional displacement vector and q is an unknown function of the coordinates. Let us write the displacement vector $\mathbf{w}$ and the tensor $\boldsymbol{\Theta}$ in the basis $\mathbf{e}_R, \mathbf{e}_\Phi$, in their component form

$$\mathbf{w} = u\mathbf{e}_R + v\mathbf{e}_\Phi; \quad \boldsymbol{\Theta} = \theta_{RR}\mathbf{e}_R\mathbf{e}_R + \theta_{R\Phi}\mathbf{e}_R\mathbf{e}_\Phi + \ldots$$

Then the equations of neutral equilibrium (2.1) written in polar coordinates will take the form

$$\frac{\partial \theta_{RR}}{\partial R} + \frac{\theta_{RR} - \theta_{\Phi\Phi}}{R} + \frac{\partial \theta_{\Phi R}}{R \partial \Phi} = 0, \quad \frac{\partial \theta_{R\Phi}}{\partial R} + \frac{\theta_{R\Phi} + \theta_{\Phi R}}{R} + \frac{\partial \theta_{\Phi\Phi}}{R \partial \Phi} = 0 \tag{2.2}$$
$$\frac{\partial u}{\partial R} + \frac{u}{R} + \frac{\partial v}{R \partial \Phi} = 0$$

The last equation in (2.2) represents the condition of incompressibility. Using (1.2) and (1.4) we obtain the following expressions for the components of the tensor $\boldsymbol{\Theta}$ in terms of the displacement vector components:

$$\theta_{RR} = \varkappa_1 \left[\frac{\partial u}{\partial R} + \frac{\psi}{R}\left(\frac{\partial u}{\partial \Phi} - v\right)\right] + p\frac{\partial u}{\partial R} + q \tag{2.3}$$
$$\theta_{R\Phi} = \varkappa_1 \left[\frac{\partial v}{\partial R} + \frac{\psi}{R}\left(\frac{\partial v}{\partial \Phi} + u\right)\right] + \frac{p}{R}\left(\frac{\partial u}{\partial \Phi} - v\right)$$
$$\theta_{\Phi R} = \varkappa_1 \left[\psi\frac{\partial u}{\partial R} + \frac{\psi^2 + 1}{R}\left(\frac{\partial u}{\partial \Phi} - v\right)\right] + p\frac{\partial v}{\partial R}$$
$$\theta_{\Phi\Phi} = \varkappa_1 \left[\psi\frac{\partial v}{\partial R} + \frac{\psi^2 + 1}{R}\left(\frac{\partial v}{\partial \Phi} + u\right)\right] + \frac{p}{R}\left(\frac{\partial v}{\partial \Phi} + u\right) + q$$

Substituting (2.3) into (2.2) we obtain a system of partial differential equations in u, v, q. The system clearly has solutions of the form

$$u = U(R) e^{i\alpha\Phi}, \quad v = V(R) e^{i\alpha\Phi}, \quad q = \varkappa_1 Q(R) e^{i\alpha\Phi} \tag{2.4}$$

Since the solution is periodic with respect to the angular coordinate, α is an integer. Solutions (2.4) enable us to reduce the system of partial differential equations to a single, fourth-order ordinary differential equation

$$U^{(IV)} + U'''(6R^{-1} - i4C_1\alpha R^{-3}) + U''[R^{-2}(5 - 2\alpha^2) - 4\alpha^2 C_1^2 R^{-6}] + \tag{2.5}$$
$$U'[i4C_1\alpha R^{-5}(\alpha^2 - 1) - R^{-3}(1 + 2\alpha^2) + 12\alpha^2 C_1^2 R^{-7}] +$$
$$U[\alpha^2(\alpha^2 - 1)(4C_1^2 R^{-8} + R^{-4}) + (1 - \alpha^2)(R^{-4} + i4\alpha C_1 R^{-6})] = 0$$

Using the formula for varying the oriented plane /1/ and relation (1.1), we obtain the boundary conditions at the outer ring contour

$$\theta_{RR} = \frac{\tau}{R}\left(\frac{\partial u}{\partial \Phi} - v\right), \quad \theta_{R\Phi} = -\tau\frac{\partial u}{\partial R}, \quad R = 1 + h \tag{2.6}$$

Taking into account (2.3), (2.4), (1.8), we can write (2.6) in the form

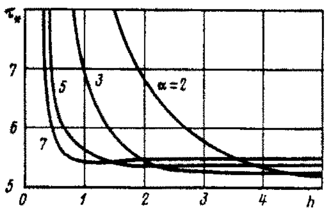
$$U'' + U'\{(1+h)^{-1} - i\alpha[\tau(1+h)^{-1} + 2C_1(1+h)^{-3}]\} + \tag{2.7}$$
$$U(\alpha^2 - 1)(1+h)^{-2} = 0$$
$$U''' + U'\{-3(\alpha^2 + 1)(1+h)^{-2} + 4\alpha^2[C_1\tau(1+h)^{-4} + C_1^2(1+$$
$$h)^{-6}] + i\alpha[10C_1(1+h)^{-4} + 3\tau(1+h)^{-2}]\} + U\{3(1 -$$
$$\alpha^2)(1+h)^{-3} + i\alpha(\alpha^2 - 1)[6C_1(1+h)^{-5} + \tau(1+h)^{-3}]\} = 0$$

and the conditions of rigid clamping along the inner contour $R = 1$ can be written as

$$U = U' = 0 \tag{2.8}$$

Thus we have reduced the problem of stability to that of solving a boundary value problem for (2.5) with boundary conditions (2.7) and (2.8). A numerical method resembling that used in /2/ yields the critical values of the tangential lead intensity τ_* values of the relative shell thickness h and the parameter α.

Direct substitution shows that the boundary value problem for the ordinary differential equations is selfconjugate. This is ensured by the existence of a potential for the stresses and the potential character of the external loads. In the case in question the latter condition holds irrespective of the follower character of the load. It can be shown that a uniform tangential load of the type (1.1) distributed over a closed contour is conservative, and this justifies the use of the static Euler method.



The figure shows the results of computations for the neo-Hookean material. The curves corresponding to various values of α characterize the dependence of τ_* on h. We see that on reducing the ring thickness the value of the critical load increases for fixed α. Thus the ring becomes more stable when h decreases. When the load is increased in steps, a form of equilibrium characterized by large values of α occurs within the range of small thicknesses. For thick rings the converse is true. We note that the stability curves have a minimum. On passing the minimum point the loads increase slightly and have a horizontal asymptote as $h \to \infty$.

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EFFECT OF A SMALL DEVIATION IN THE FORM OF THE SHELLS OF REVOLUTION FROM AXIAL SYMMETRY ON THEIR STATE OF STRESS*

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The effect of small, non-axially symmetric imperfections of the middle surface in shells of revolution on their stresses and displacements is studied. A strong dependence on them is found both in statics as well as thermoelasticity. The general theoretical results are confirmed by a numerical study of the displacement and stress fields in cylindrical and conical shells with small imperfections of the type $f(x)\cos m\varphi$.

1. We know that shells with free boundaries (we shall call them free shells) are in, general, compliant, and only weakly resist the action of external loads. However, if the external loads satisfy prescribed integral conditions formulated in the theorem on flexure /1, 2/, the shells become stiff. The stiffness is, however, unstable and vanishes when there are minute deviations from the conditions, whereupon the shell bends and large displacements result. Using the static-geometrical analogy, we find that the problem of analyzing a free shell under external load is equivalent to the problem of computing a shell clamped along its boundary (we shall call it the clamped shell) in a temperature field /3/. From this, we find that, according to the above analogy, the appearance of instability of the stress state in free shells when there are small changes in external load, implies a certain instability in the stress state in clamped shells for small variations in the temperature load.

We will derive asymptotic estimates which will be needed later, for the stress state in free and clamped shells under the action of slowly varying loads of single intensity /4/. In a free shell, when the conditions of the theorem on flexures do not hold, the tangential stress σ_1 and the flexural stress σ_2 are of the order of

$$2Eh\,(u_1,\ u_2,\ w) = h_*^{-2}a^2O\,(\mathbf{R});\ \sigma_1 = h_*^{-1}O\,(\mathbf{R});\ \sigma_2 = h_*^{-2}O\,(\mathbf{R})$$

Here u_1, u_2, w are the displacement vector components, h is the half-thickness of the shell, $\mathbf{R}$ is the external load vector, a is the characteristic linear dimension, E is Young's modulus and $h_* = h/a$ is a small parameter. In a clamped shell we have

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